Uitwerkingen Tentamen Wiskunde VI, 17-juni-1993.

1(a) Stel $v(x,y) = \operatorname{Im} f(x+iy)$. f geheel $\Rightarrow 0 = \Delta v = 2+2a \Rightarrow \boxed{a=-1}$

Stel $f = u+iv$. f geheel $\Rightarrow$ Cauchy Riemann:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -2y-1 \Rightarrow u = -2xy - x + C(y) \Rightarrow -2x + C'(y) = \frac{\partial u}{\partial y} =$$

$$= -\frac{\partial v}{\partial x} = -2x-1 \Rightarrow C'(y) = -1 \Rightarrow C(y) = -y + k \Rightarrow$$

$$f(x+iy) = u(x,y)+iv(x,y) = -2xy - x - y + k + i(x^2+x-y^2-y),\ k\in\mathbb{R}$$

$$f(x+i0) = -x+k+i(x^2+x) \xRightarrow{\text{identiteitsst.}} \boxed{f(z) = -z+k+i(z^2+z),\ k\in\mathbb{R}}$$

(b) $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \Rightarrow a_m = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z^{m+1}}\,dz$, $z\neq 0$

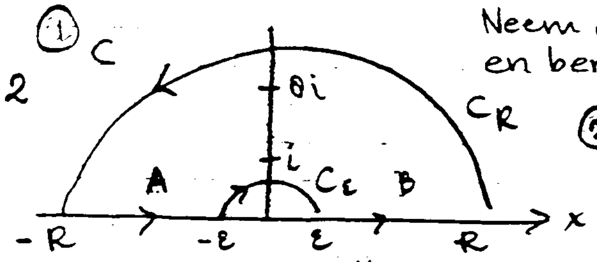
Hier: $f(z) = (z^2+z^{-2})^3 = z^6 + 3z^2 + 3z^{-2} + z^{-6} \Rightarrow a_m = \begin{cases}1 & m=\pm 6\\ 3 & m=\pm 2\\ 0 & \text{andere } m\text{'s}\end{cases} \in\mathbb{R}!$

en
$$a_m = \frac{1}{2\pi i}\int_{|z|=1}\frac{(z^2+z^{-2})^3}{z^{m+1}}\,dz \underset{z=e^{i\varphi}}{=} \frac{1}{2\pi i}\int_0^{2\pi}\frac{(2\cos 2\varphi)^3}{e^{i(m+1)\varphi}}\cdot ie^{i\varphi}\,d\varphi$$

$$= \frac{4}{\pi}\int_0^{2\pi}(\cos 2\varphi)^3 e^{-im\varphi}\,d\varphi = \operatorname{Re}\ldots = \frac{4}{\pi}\int_0^{2\pi}(\cos 2\varphi)^3\cos m\varphi\,d\varphi$$

$$\Rightarrow \int_0^{2\pi}(\cos 2\varphi)^3\cos m\varphi\,d\varphi = \begin{cases}\pi/4, & m=\pm 6\\ 3\pi/4 & m=\pm 2\\ 0 & \text{andere } m\text{'s}\end{cases}$$

2 ①



Neem als contour C en bereken I via $\int_C \frac{z^{-1/3}}{(z^2+1)(z^2+64)}\,dz$:

② $$\operatorname{Res}_{z=i=e^{\frac12\pi i}} \frac{z^{-1/3}}{(z+i)(z-i)(z^2+64)} = \frac{e^{-\frac16\pi i}}{2i\cdot 63}$$

$$\operatorname{Res}_{z=8i=2^3e^{\frac12\pi i}} \frac{z^{-1/3}}{(z^2+1)(z+8i)(z-8i)} = \frac{\frac12 e^{-\frac16\pi i}}{-63\cdot 16i}$$

som residuen $= \frac{\cancel{15}\ 5\, e^{-\frac16\pi i}}{2\cdot 16\cdot \cancel{63}\ 21\, i}$

③ $$\left|\int_{C_R}\right| \le \frac{R^{-1/3}}{(R^2-1)(R^2-64)}\pi R \to 0,\quad R\to\infty$$

$$\left|\int_{C_\varepsilon}\right| \le \frac{\varepsilon^{-1/3}\pi\varepsilon}{(1-\varepsilon^2)(64-\varepsilon^2)} \to 0,\quad \varepsilon\downarrow 0$$

$$\int_A \underset{z=xe^{\pi i}}{=} \int_R^\varepsilon \frac{x^{-1/3}e^{-1/3\pi i}\,\overbrace{e^{\pi i}}^{=-1}dx}{(x^2\underbrace{e^{2\pi i}}_{=1}+1)(x^2\underbrace{e^{2\pi i}}_{=1}+64)} = \int_\varepsilon^R \frac{x^{-1/3}e^{-1/3\pi i}}{(x^2+1)(x^2+64)}\,dx \underset{R\to\infty,\ \varepsilon\downarrow 0}{\to} e^{-\frac13\pi i} I$$

$$\int_B \underset{z=x}{=} \int_\varepsilon^R \frac{x^{-1/3}}{(x^2+1)(x^2+64)}\,dx \to I\ (R\to\infty,\ \varepsilon\downarrow 0)$$

④ $$\int_C = \int_{C_R}+\int_{C_\varepsilon}+\int_A+\int_B = \frac{\cancel{2\pi i}\ 5e^{-\frac16\pi i}}{\cancel{2}\cdot 16\cdot 21\,\cancel{i}} = \frac{5\pi e^{-\frac16\pi i}}{16\cdot 21}$$

Laat $R\to\infty$, $\varepsilon\downarrow 0 \Rightarrow I(1+e^{-\frac13\pi i}) = \frac{5\pi e^{-\frac16\pi i}}{16\cdot 21} \Rightarrow I = \frac{5\pi}{16\cdot 21\,(e^{\frac16\pi i}+e^{-\frac16\pi i})} = \frac{5\pi}{16\cdot 21\cdot 2\cos\frac{\pi}{6}}$

$$= \frac{5\pi}{16\cdot 21\sqrt3} = \frac{5\pi\sqrt3}{1008}.$$